Lecture 7 : Borel-Cantelli lemmas and almost sure convergence

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This set of notes is a revision of the work of Jin Kim, 2002.

7.1 Borel-Cantelli Lemmas

Recall that for real valued random variables X_n and X,

$$\begin{aligned}\{X_n \to X\} &= \{\omega : X_n(\omega) \to X(\omega)\} \\ &= \{\forall \epsilon > 0, |X_n - X| \leq \epsilon \ \text{ eventually}\}\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{P}(X_n \to X) = 1 &\Leftrightarrow \forall \epsilon > 0,\ \mathbb{P}(|X_n - X| \leq \epsilon \ ev.) = 1 \\ &\Leftrightarrow \forall \epsilon > 0,\ \mathbb{P}(|X_n - X| > \epsilon \ i.o.) = 0\end{aligned}$$

Let the event $A_n := \{|X_n - X| > \epsilon\}$. Then, we are motivated by consideration of a.s. convergence to find useful conditions for $\mathbb{P}(A_n \text{ i.o.}) = 0$.

Recall that $\{A_n \text{ i.o.}\} = \bigcap_n \bigcup_{m \geq n} A_m$.

Theorem 7.1 (Borel-Cantelli Lemmas) *Let $(\Omega, F, \mathbb{P})$ be a probability space and let (A_n) be a sequence of events in F. Then,*

1. *If $\sum_n \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(A_n \ i.o.) = 0$.*

2. *If $\sum_n \mathbb{P}(A_n) = \infty$ and A_n are independent, then $\mathbb{P}(A_n \ i.o.) = 1$.*

There are many possible substitutes for independence in BCL II, see Kochen-Stone Lemma.

Before proving BCL, notice that

- $\mathbf{1}(A_n \text{ i.o.}) = \limsup_{n\to\infty} \mathbf{1}(A_n)$

- $\mathbf{1}(A_n \text{ ev.}) = \liminf_{n\to\infty} \mathbf{1}(A_n)$
- $\{A_n \text{ i.o.}\} = \lim_{m\to\infty}(\cup_{n>m} A_n)$ (note: as $m \uparrow$, $\cup_{n\geq m} A_n \downarrow$)
- $\{A_n \text{ ev.}\} = \lim_{m\to\infty}(\cap_{n>m} A_n)$ (note: as $m \uparrow$, $\cap_{n\geq m} A_n \uparrow$).

Therefore,

$$\begin{aligned}
\mathbb{P}(A_n \text{ ev.}) &\leq \liminf_{n\to\infty} \mathbb{P}(A_n) && \text{by Fatou's lemma} \\
&\leq \limsup_{n\to\infty} \mathbb{P}(A_n) && \text{obvious from definition} \\
&\leq \mathbb{P}(A_n \text{ i.o.}) && \text{dual of Fatou's lemma (i.e. apply to } -\mathbb{P})
\end{aligned}$$

Proof: (Of BCL I)

$$\begin{aligned}
\mathbb{P}(A_n \text{ i.o.}) &= \lim_{m\to\infty} \mathbb{P}(\cup_{n\geq m} A_n) \\
&\leq \lim_{m\to\infty} \sum_{n\geq m}^{\infty} \mathbb{P}(A_n) \;=\; 0 \quad \text{since } \sum_{i=1}^{\infty} \mathbb{P}(A_n) < \infty.
\end{aligned}$$

■

Proof: (Of BCL II) Assume that $\Sigma\mathbb{P}(A_n) = \infty$ and the A_n's are independent. We will show that $\mathbb{P}(A_n^c \text{ ev.}) = 0$.

$$\begin{aligned}
\mathbb{P}(A_n^c \text{ ev.}) &= \lim_{n\to\infty} \mathbb{P}(\cap_{m\geq n} A_m^c) \;=\; \lim_{n\to\infty} \prod_{m\geq n} \mathbb{P}(A_m^c) && (7.1) \\
&= \lim_{n\to\infty} \prod_{m\geq n} (1 - \mathbb{P}(A_m)) \;\leq\; \lim_{n\to\infty} \prod_{m\geq n} \exp\left(-\mathbb{P}(A_m^c)\right) && (7.2) \\
&= \lim_{n\to\infty} \exp\left(-\sum_{m\geq n} \mathbb{P}(A_m^c)\right) = 0
\end{aligned}$$

since $(-\sum_{m\geq n} \mathbb{P}(A_m^c)) \to \infty$, as $n \to \infty$

For (7.1), we used the following fact (due to the independence of A_n):

$$\mathbb{P}(\cap_{m\geq n} A_m^c) = \lim_{N\to\infty} \mathbb{P}(\cap_{n\leq m\leq N} A_m^c) = \lim_{N\to\infty} \prod_{n\leq m\leq N} \mathbb{P}(A_m^c) = \prod_{n\leq m} \mathbb{P}(A_m^c).$$

For (7.2), $1 - x \leq \exp(-x)$ was used. ■

For an example in which the theorem cannot be applied, consider $A_n = (0, 1/n)$ in $(0, 1)$. Then, $\mathbb{P}(A_n) = 1/n$, $\sum \mathbb{P}(A_n) = \infty$, but $\mathbb{P}(A_n \text{ i.o.}) = \mathbb{P}(\emptyset) = 0$.

Example 7.2 *Consider random walk in* $\mathbb{Z}^d$, $d = 0, 1, \cdots$ $S_n = X_1 + \cdots + X_n,$, $n = 0, 1, \cdots$ *where* X_i *are independent in* $\mathbb{Z}^d$. *In the simplest case, each* X_i *has uniform distribution on* 2^d *possible strings. i.e., if* $d = 3$, *we have* $2^3 = 8$ *neighbors*

$$\left\{ \begin{array}{c} (+1, +1, +1) \\ \vdots \\ (-1, -1, -1) \end{array} \right\}.$$

Note that each coordinate of S_n *does a simple coin-tossing walk independently. We can prove that*

$$\mathbb{P}(S_n = 0 \ i.o.) = \begin{cases} 1 & \text{if } d = 1 \text{ or } 2 \quad \text{(recurrent)} \\ 0 & \text{if } d \geq 3 \qquad\quad\ \text{(transient)}. \end{cases} \tag{7.3}$$

Proof Sketch: (of (7.3))
Let us start with $d = 1$, then

$$\mathbb{P}(S_{2n} = 0) = \mathbb{P}(n \text{ “+” signs and } n \text{ “−” signs}) \tag{7.4}$$
$$= \binom{2n}{n} 2^{-2n} \tag{7.5}$$
$$\sim \frac{c}{\sqrt{n}} \text{ as } n \longrightarrow \infty. \tag{7.6}$$

where we used the facts that $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$, and that $a_n \sim b_n$ iff $\frac{a_n}{b_n} \to 1$ as $n \to \infty$.

Note

$$\sum \left(\frac{1}{\sqrt{n}}\right)^d \begin{cases} = \infty & d = 1, 2 \\ < \infty & d = 3, 4, \cdots \end{cases} \tag{7.7}$$

Thus, $\sum_n \mathbb{P}(S_{2n} = 0) = \infty$, and BC II and (7.7) together gives (7.3). ■

Example 7.3 (for the case $d = 1$**)** $\{S_2 = 0\}$ *is the event of ending up back to the origin at step 2 when we started at the origin.* $\mathbb{P}(S_2 = 0) = 1/2$. *Note:*

$$\mathbb{P}(S_{10,000} = 0) \sim \frac{c}{\sqrt{n}} \approx 1/100,$$

$$\mathbb{P}(S_{10,002} = 0) \approx 1/100,$$

$$\mathbb{P}(S_{10,000} = 0, S_{10,002} = 0) = \mathbb{P}(S_{10,000} = 0)\mathbb{P}(S_{10,002} = 0 | S_{10,000} = 0) \approx 1/100 \cdot 1/2,$$

Later in the course, we will show that for the case $d = 1$, *even when the* $(S_{2n} = 0)$ *are dependent, it is still true that* $\mathbb{P}(S_{2n} = 0 \ i.o. \ ) = 1$.

The same result holds for the case $d = 2$.

In general,

$$\mathbb{P}(S_{2n} = 0) = \frac{\binom{2n}{n}^d}{2^{2n}} \approx \frac{c^d}{n^{d/2}}.$$

For $d = 2$, this is $\sim \frac{c^2}{n}$ which is not summable. Thus, $\mathbb{P}(S_{2n} = 0 \text{ i.o.}) = 1$. For $d \geq 3$, this is $\sim \frac{c^3}{n^{3/2}}$ which is summable. Then, by **BCL I**, $\mathbb{P}(S_{2n} = 0 \text{ i.o.}) = 0$.

7.2 Almost sure convergence

Because

$$X_n \longrightarrow X \text{ a.s.} \qquad \Longleftrightarrow \qquad X_n - X \longrightarrow 0 \text{ a.s.} ,$$

it is enough to prove for the case of convergence to 0.

Proposition 7.4 *The following are equivalent:*

1. $X_n \xrightarrow{a.s.} 0$
2. $\forall \epsilon > 0,\ \mathbb{P}(|X_n| > \epsilon \ i.o.) = 0$
3. $M_n \xrightarrow{\mathbb{P}} 0$ *where* $M_n := \sup_{n \leq k} |X_k|$
4. $\forall\ \epsilon_n \downarrow 0\ :\ \mathbb{P}(|X_n| > \epsilon_n\ i.o.) = 0$

Note: "$\forall$" in Proposition 4 cannot be replaced by "$\exists$". For example, Let $X_n = (1/\sqrt{n})U_n$, where $U_1, U_2, ...$ are independent $U[0,1]$.

Take $\epsilon_n = 1/2/\sqrt{n}$. Then, $\mathbb{P}(X_n > \epsilon_n) = \mathbb{P}(U_n > 1/2) = 1/2$. So, $\mathbb{P}(X_n > \epsilon_n \text{ i.o.}) = 1$.

But if we take $\epsilon_n = \frac{1}{\sqrt{n}}$. Then, $\mathbb{P}(X_n > \epsilon_n) = \mathbb{P}(U_n > 1) = 0$.

Proof: (only for the equivalence of 1 and 3)

Suppose Proposition 1 holds. If $X_n(\omega) \to 0$ a.s., then $\sup_{n \leq k} |X_k(\omega)| \to 0$ a.s. But this implies that $M_n \to 0$ a.s. Thus, $M_n \xrightarrow{\mathbb{P}} 0$.

Conversely, if $M_n \downarrow$ as $n \uparrow$, then we know in advance that M_n has a almost-surely-limit in $[0, \infty]$. ■

Lemma 7.5 *If $X_n \xrightarrow{\mathbb{P}} X$, then there exists a subsequence n_k such that $X_{n_k} \to X$ a.s.*

Proof: It is enough to show that there exists $\epsilon_k \downarrow 0$ such that $\sum_k \mathbb{P}(|X_{n_k} - X| > \epsilon_k) < \infty$. We can take $\epsilon_k = 1/k$ and choose n_k so that $\mathbb{P}(|X_{n_k} - X| > 1/k) \leq 1/2^k$. Then, $\sum_k \mathbb{P}(|X_{n_k} - X| > \epsilon_k) < \infty$, and by **BCL I** we can conclude that $X_{n_k} \to X$ a.s. ■